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ROUTLEDGE

NINTH EDITION

BIRD'S HIGHER ENGINEERING MATHEMATICS

JOHN BIRD

Bird's Higher Engineering Mathematics

Why is knowledge of mathematics important in engineering?

A career in any engineering or scientific field will require both basic and advanced mathematics. Without mathematics to determine principles, calculate dimensions and limits, explore variations, prove concepts and so on, there would be no mobile telephones, televisions, stereo systems, video games, microwave ovens, computers or virtually anything electronic. There would be no bridges, tunnels, roads, skyscrapers, automobiles, ships, planes, rockets or most things mechanical. There would be no metals beyond the common ones, such as iron and copper, no plastics, no synthetics. In fact, society would most certainly be less advanced without the use of mathematics throughout the centuries and into the future.

Electrical engineers require mathematics to design, develop, test or supervise the manufacturing and installation of electrical equipment, components or systems for commercial, industrial, military or scientific use.

Mechanical engineers require mathematics to perform engineering duties in planning and designing tools, engines, machines and other mechanically functioning equipment; they oversee installation, operation, maintenance and repair of such equipment as centralised heat, gas, water and steam systems.

Aerospace engineers require mathematics to perform a variety of engineering work in designing, constructing and testing aircraft, missiles and spacecraft; they conduct basic and applied research to evaluate adaptability of materials and equipment to aircraft design and manufacture and recommend improvements in testing equipment and techniques.

Nuclear engineers require mathematics to conduct research on nuclear engineering problems or apply principles and theory of nuclear science to problems concerned with release, control and utilisation of nuclear energy and nuclear waste disposal.

Petroleum engineers require mathematics to devise methods to improve oil and gas well production and determine the need for new or modified tool designs; they oversee drilling and offer technical advice to achieve economical and satisfactory progress.

Industrial engineers require mathematics to design, develop, test and evaluate integrated systems for managing industrial production processes, including human work factors, quality control, inventory control, logistics and material flow, cost analysis and production co-ordination.

Environmental engineers require mathematics to design, plan or perform engineering duties in the prevention, control and remediation of environmental health hazards, using various engineering disciplines; their work may include waste treatment, site remediation or pollution control technology.

Civil engineers require mathematics in all levels in civil engineering – structural engineering, hydraulics and geotechnical engineering are all fields that employ mathematical tools such as differential equations, tensor analysis, field theory, numerical methods and operations research.

Knowledge of mathematics is therefore needed by each of the engineering disciplines listed above.

It is intended that this text – *Bird's Higher Engineering Mathematics* – will provide a step-by-step approach to learning the essential mathematics needed for your engineering studies.

Now in its ninth edition, *Bird's Higher Engineering Mathematics* has helped thousands of students to succeed in their exams. Mathematical theories are explained in a straightforward manner, supported by practical engineering examples and applications to ensure that readers can relate theory to practice. Some 1,200 engineering situations/problems have been 'flagged-up' to help demonstrate that engineering cannot be fully understood without a good knowledge of mathematics.

The extensive and thorough topic coverage makes this an ideal text for undergraduate degree courses, foundation degrees, and for higher-level vocational courses such as Higher National Certificate and Diploma courses in engineering disciplines.

Its companion website at **www.routledge.com/cw/bird** provides resources for both students and lecturers, including full solutions for all 2,100 further questions, lists of essential formulae, multiple-choice tests, and illustrations, as well as full solutions to revision tests for course instructors.

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Bird's Higher Engineering Mathematics

Ninth Edition

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To Sue

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Preface

This **ninth edition of '*Bird's Higher Engineering Mathematics'*** covers essential mathematical material suitable for students studying **Degrees, Foundation Degrees and Higher National Certificate and Diploma courses in Engineering disciplines.**

The text has been conveniently divided into the following **thirteen convenient categories**: number and algebra, geometry and trigonometry, graphs, complex numbers, matrices and determinants, vector geometry, differential calculus, integral calculus, differential equations, Laplace transforms, Fourier series, Z-transforms and statistics and probability.

Increasingly, **difficulty in understanding algebra** is proving a problem for many students as they commence studying engineering courses. Inevitably there are a lot of formulae and calculations involved with engineering studies that require a sound grasp of algebra. On the website, **www.routledge.com/bird** is a document which offers **a quick revision of the main areas of algebra** essential for further study, i.e. basic algebra, simple equations, transposition of formulae, simultaneous equations and quadratic equations.

In this new edition, all but three of the chapters of the previous edition are included (those excluded can be found in *Bird's Engineering Mathematics 8th Edition* or on the website), but the order of presenting some of the chapters has been changed. Problems where ***engineering applications*** occur have been 'flagged up' and some multiple-choice questions have been added to many of the chapters.

The **primary aim of the material in this text** is to provide the fundamental analytical and underpinning knowledge and techniques needed to successfully complete scientific and engineering principles modules of Degree, Foundation Degree and Higher National Engineering programmes. The material has been designed to enable students to use techniques learned for the analysis, modelling and solution of realistic engineering problems at Degree and Higher National level. It also aims to provide some of the more advanced knowledge required for those wishing to pursue careers in mechanical engineering, aeronautical engineering, electrical and electronic engineering, communications engineering, systems engineering and all variants of control engineering.

In ***Bird's Higher Engineering Mathematics 9th Edition***, theory is introduced in each chapter by a full outline of essential definitions, formulae, laws, procedures etc; **problem solving** is extensively used to establish and exemplify the theory. It is intended that readers will gain real understanding through seeing problems solved and then through solving similar problems themselves.

Access to the plethora of **software packages**, or a graphics calculator, will enhance understanding of some of the topics in this text.

Each topic considered in the text is presented in a way that assumes in the reader only knowledge attained in BTEC National Certificate/Diploma, or similar, in an Engineering discipline.

***'Birds Higher Engineering Mathematics 9th Edition'* provides a follow-up to '*Bird's Engineering Mathematics 9th Edition'*.**

This textbook contains over **1100 worked problems**, followed by some **2100 further problems (with answers),** arranged within **317 Practice Exercises**. Some **450 multiple-choice questions** are also included, together with **573 line diagrams** to further enhance understanding.

Worked solutions to all 2100 of the further problems have been prepared and can be **accessed free by students and staff via the website www.routledge.com/bird**

Where at all possible, the problems mirror practical situations found in engineering and science. In fact, some **1200 engineering situations/problems** have been 'flagged-up' to help demonstrate that engineering cannot be fully understood without a good knowledge of mathematics. Look out for the symbol

At the end of the text, a list of **Essential Formulae** is included for convenience of reference.

At intervals throughout the text are some **20 Revision Tests** to check understanding. For example, Revision Test 1 covers the material in chapters 1 to 4, Revision Test 2 covers the material in chapters 5 to 7, Revision Test 3 covers the material in chapters 8 to 10, and so on. Full solutions to the 20 Revision Tests are available free to lecturers on the website **www.routledge.com/cw/bird**

'Learning by example' is at the heart of '*Bird's Higher Engineering Mathematics 9th Edition*'.

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Free Web downloads are available at www.routledge.com/cw/bird

For Students

1. **Full solutions** to the 2100 questions contained in the 317 Practice Exercises
2. Revision of some important algebra topics
3. **List of Essential Formulae**
4. **Famous Engineers/Scientists** – 32 are mentioned in the text.
5. **Copies of chapters from the previous edition that have been excluded from this text** (these being: 'Inequalities', 'Arithmetic and geometric progressions', and 'Binary, octal and hexadecimal numbers')

For instructors/lecturers

1. **Full solutions** to the 2100 questions contained in the 317 Practice Exercises
2. **Full solutions** and marking scheme to each of the **20 Revision Tests**
3. **Revision Tests – available to run off to be given to students**
4. **List of Essential Formulae**
5. **Illustrations – all 573 available on PowerPoint**
6. **Famous Engineers/Scientists** – 32 are mentioned in the text

Syllabus guidance

This textbook is written for **undergraduate engineering degree and foundation degree courses**; however, it is also most appropriate for **HNC/D studies** and three syllabuses are covered. The appropriate chapters for these three syllabuses are shown in the table below.

Chapter		**Analytical Methods for Engineers**	**Further Analytical Methods for Engineers**	**Advanced Mathematics**
1.	Algebra	×		
2.	Partial fractions	×		
3.	Logarithms	×		
4.	Exponential functions	×		
5.	The binomial series	×		
6.	Solving equations by iterative methods		×	
7.	Boolean algebra and logic circuits		×	
8.	Introduction to trigonometry	×		
9.	Cartesian and polar co-ordinates	×		
10.	The circle and its properties	×		
11.	Trigonometric waveforms	×		
12.	Hyperbolic functions	×		
13.	Trigonometric identities and equations	×		
14.	The relationship between trigonometric and hyperbolic functions	×		
15.	Compound angles	×		
16.	Functions and their curves		×	
17.	Irregular areas, volumes and mean value of waveforms		×	
18.	Complex numbers		×	
19.	De Moivre's theorem		×	
20.	The theory of matrices and determinants		×	
21.	The solution of simultaneous equations by matrices and determinants		×	
22.	Vectors		×	
23.	Methods of adding alternating waveforms		×	
24.	Scalar and vector products		×	
25.	Methods of differentiation	×		

(*Continued*)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Advanced Mathematics for Engineering
26.	Some applications of differentiation	×		
27.	Differentiation of parametric equations			
28.	Differentiation of implicit functions	×		
29.	Logarithmic differentiation	×		
30.	Differentiation of hyperbolic functions	×		
31.	Differentiation of inverse trigonometric and hyperbolic functions	×		
32.	Partial differentiation			×
33.	Total differential, rates of change and small changes			×
34.	Maxima, minima and saddle points for functions of two variables			×
35.	Standard integration	×		
36.	Some applications of integration	×		
37.	Maclaurin's series and limiting values	×		
38.	Integration using algebraic substitutions	×		
39.	Integration using trigonometric and hyperbolic substitutions	×		
40.	Integration using partial fractions	×		
41.	The t = tan $\theta/2$ substitution			
42.	Integration by parts	×		
43.	Reduction formulae	×		
44.	Double and triple integrals			
45.	Numerical integration		×	
46.	Solution of first-order differential equations by separation of variables		×	
47.	Homogeneous first-order differential equations			
48.	Linear first-order differential equations		×	
49.	Numerical methods for first-order differential equations		×	×
50.	Second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$		×	
51.	Second-order differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$		×	
52.	Power series methods of solving ordinary differential equations			×
53.	An introduction to partial differential equations			×

(Continued)

Chapter		Analytical Methods for Engineers	Further Analytical Methods for Engineers	Advanced Mathematics for Engineering
54.	Introduction to Laplace transforms			×
55.	Properties of Laplace transforms			×
56.	Inverse Laplace transforms			×
57.	The Laplace transform of the Heaviside function			
58.	Solution of differential equations using Laplace transforms			×
59.	The solution of simultaneous differential equations using Laplace transforms			×
60.	Fourier series for periodic functions of period 2π			×
61.	Fourier series for non-periodic functions over range 2π			×
62.	Even and odd functions and half-range Fourier series			×
63.	Fourier series over any range			×
64.	A numerical method of harmonic analysis			×
65.	The complex or exponential form of a Fourier series			×
66.	An introduction to z-transforms			
67.	Presentation of statistical data	×		
68.	Mean, median, mode and standard deviation	×		
69.	Probability	×		
70.	The binomial and Poisson distributions	×		
71.	The normal distribution	×		
72.	Linear correlation	×		
73.	Linear regression	×		
74.	Sampling and estimation theories	×		
75.	Significance testing	×		
76.	Chi-square and distribution-free tests	×		

Section A

Number and algebra

Chapter 1

Algebra

Why it is important to understand: **Algebra, polynomial division and the factor and remainder theorems**

It is probably true to say that there is no branch of engineering, physics, economics, chemistry or computer science which does not require the understanding of the basic laws of algebra, the laws of indices, the manipulation of brackets, the ability to factorise and the laws of precedence. This then leads to the ability to solve simple, simultaneous and quadratic equations which occur so often. The study of algebra also revolves around using and manipulating polynomials. Polynomials are used in engineering, computer programming, software engineering, in management and in business. Mathematicians, statisticians and engineers of all sciences employ the use of polynomials to solve problems; among them are aerospace engineers, chemical engineers, civil engineers, electrical engineers, environmental engineers, industrial engineers, materials engineers, mechanical engineers and nuclear engineers. The factor and remainder theorems are also employed in engineering software and electronic mathematical applications, through which polynomials of higher degrees and longer arithmetic structures are divided without any complexity. The study of algebra, equations, polynomial division and the factor and remainder theorems is therefore of some considerable importance in engineering.

At the end of this chapter, you should be able to:

- understand and apply the laws of indices
- understand brackets, factorisation and precedence
- transpose formulae and solve simple, simultaneous and quadratic equations
- divide algebraic expressions using polynomial division
- factorise expressions using the factor theorem
- use the remainder theorem to factorise algebraic expressions

1.1 Introduction

In this chapter, polynomial division and the factor and remainder theorems are explained (in Sections 1.4 to 1.6). However, before this, some essential algebra revision on basic laws and equations is included.
For further algebra revision, go to the website:

www.routledge.com/cw/bird

1.2 Revision of basic laws

(a) Basic operations and laws of indices

The **laws of indices** are:

(i) $a^m \times a^n = a^{m+n}$ (ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{m\times n}$ (iv) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$ (vi) $a^0 = 1$

Problem 1. Evaluate $4a^2bc^3 - 2ac$ when $a = 2$, $b = \frac{1}{2}$ and $c = 1\frac{1}{2}$

$$4a^2bc^3 - 2ac = 4(2)^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^3 - 2(2)\left(\frac{3}{2}\right)$$

$$= \frac{4 \times 2 \times 2 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2} - \frac{12}{2}$$

$$= 27 - 6 = \mathbf{21}$$

Problem 2. Multiply $3x + 2y$ by $x - y$

$$\begin{array}{lr} & 3x + 2y \\ & x - y \\ \hline \text{Multiply by } x \rightarrow & 3x^2 + 2xy \\ \text{Multiply by } -y \rightarrow & -3xy - 2y^2 \\ \hline \text{Adding gives:} & \mathbf{3x^2 - xy - 2y^2} \\ \hline \end{array}$$

Alternatively,

$$(3x + 2y)(x - y) = 3x^2 - 3xy + 2xy - 2y^2$$

$$= \mathbf{3x^2 - xy - 2y^2}$$

Problem 3. Simplify $\frac{a^3b^2c^4}{abc^{-2}}$ and evaluate when $a = 3$, $b = \frac{1}{8}$ and $c = 2$

$$\frac{a^3b^2c^4}{abc^{-2}} = a^{3-1}b^{2-1}c^{4-(-2)} = \boldsymbol{a^2bc^6}$$

When $a = 3$, $b = \frac{1}{8}$ and $c = 2$,

$$a^2bc^6 = (3)^2\left(\frac{1}{8}\right)(2)^6 = (9)\left(\frac{1}{8}\right)(64) = \mathbf{72}$$

Problem 4. Simplify $\frac{x^2y^3 + xy^2}{xy}$

$$\frac{x^2y^3 + xy^2}{xy} = \frac{x^2y^3}{xy} + \frac{xy^2}{xy}$$

$$= x^{2-1}y^{3-1} + x^{1-1}y^{2-1}$$

$$= \boldsymbol{xy^2 + y} \quad \text{or} \quad \boldsymbol{y(xy + 1)}$$

Problem 5. Simplify $\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}}$

$$\frac{(x^2\sqrt{y})(\sqrt{x}\sqrt[3]{y^2})}{(x^5y^3)^{\frac{1}{2}}} = \frac{x^2y^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{2}{3}}}{x^{\frac{5}{2}}y^{\frac{3}{2}}}$$

$$= x^{2+\frac{1}{2}-\frac{5}{2}}y^{\frac{1}{2}+\frac{2}{3}-\frac{3}{2}}$$

$$= x^0y^{-\frac{1}{3}}$$

$$= \boldsymbol{y^{-\frac{1}{3}}} \quad \text{or} \quad \boldsymbol{\frac{1}{y^{\frac{1}{3}}}} \quad \text{or} \quad \boldsymbol{\frac{1}{\sqrt[3]{y}}}$$

Now try the following Practice Exercise

Practice Exercise 1 Basic algebraic operations and laws of indices (Answers on page 863)

1. Evaluate $2ab + 3bc - abc$ when $a = 2$, $b = -2$ and $c = 4$
2. Find the value of $5pq^2r^3$ when $p = \frac{2}{5}$, $q = -2$ and $r = -1$
3. From $4x - 3y + 2z$ subtract $x + 2y - 3z$
4. Multiply $2a - 5b + c$ by $3a + b$
5. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x = \frac{1}{2}$, $y = 2$ and $z = 3$
6. Evaluate $(a^{\frac{3}{2}}bc^{-3})(a^{\frac{1}{2}}b^{-\frac{1}{2}}c)$ when $a = 3$, $b = 4$ and $c = 2$
7. Simplify $\frac{a^2b + a^3b}{a^2b^2}$
8. Simplify $\frac{(a^3b^{\frac{1}{2}}c^{-\frac{1}{2}})(ab)^{\frac{1}{3}}}{(\sqrt{a^3}\sqrt{b}\,c)}$

(b) Brackets, factorisation and precedence

Problem 6. Simplify $a^2 - (2a - ab) - a(3b + a)$

$$a^2 - (2a - ab) - a(3b + a)$$

$$= a^2 - 2a + ab - 3ab - a^2$$

$$= \boldsymbol{-2a - 2ab} \quad \text{or} \quad \boldsymbol{-2a(1 + b)}$$

Problem 7. Remove the brackets and simplify the expression:

$$2a - [3\{2(4a - b) - 5(a + 2b)\} + 4a]$$

Removing the innermost brackets gives:

$$2a - [3\{8a - 2b - 5a - 10b\} + 4a]$$

Collecting together similar terms gives:

$$2a - [3\{3a - 12b\} + 4a]$$

Removing the 'curly' brackets gives:

$$2a - [9a - 36b + 4a]$$

Collecting together similar terms gives:

$$2a - [13a - 36b]$$

Removing the square brackets gives:

$$2a - 13a + 36b = \mathbf{-11a + 36b} \quad \text{or} \quad \mathbf{36b - 11a}$$

Problem 8. Factorise (a) $xy - 3xz$ (b) $4a^2 + 16ab^3$ (c) $3a^2b - 6ab^2 + 15ab$

(a) $xy - 3xz = \mathbf{x(y - 3z)}$

(b) $4a^2 + 16ab^3 = \mathbf{4a(a + 4b^3)}$

(c) $3a^2b - 6ab^2 + 15ab = \mathbf{3ab(a - 2b + 5)}$

Problem 9. Simplify $3c + 2c \times 4c + c \div 5c - 8c$

The order of precedence is division, multiplication, addition, and subtraction (sometimes remembered by BODMAS). Hence

$$\begin{aligned} &3c + 2c \times 4c + c \div 5c - 8c \\ &= 3c + 2c \times 4c + \left(\frac{c}{5c}\right) - 8c \\ &= 3c + 8c^2 + \frac{1}{5} - 8c \\ &= \mathbf{8c^2 - 5c + \frac{1}{5}} \quad \text{or} \quad \mathbf{c(8c - 5) + \frac{1}{5}} \end{aligned}$$

Problem 10. Simplify $(2a - 3) \div 4a + 5 \times 6 - 3a$

$$\begin{aligned} &(2a - 3) \div 4a + 5 \times 6 - 3a \\ &= \frac{2a - 3}{4a} + 5 \times 6 - 3a \\ &= \frac{2a - 3}{4a} + 30 - 3a \\ &= \frac{2a}{4a} - \frac{3}{4a} + 30 - 3a \\ &= \frac{1}{2} - \frac{3}{4a} + 30 - 3a = \mathbf{30\frac{1}{2} - \frac{3}{4a} - 3a} \end{aligned}$$

Now try the following Practice Exercise

Practice Exercise 2 Brackets, factorisation and precedence (Answers on page 863)

1. Simplify $2(p + 3q - r) - 4(r - q + 2p) + p$
2. Expand and simplify $(x + y)(x - 2y)$
3. Remove the brackets and simplify:

   $24p - [2\{3(5p - q) - 2(p + 2q)\} + 3q]$
4. Factorise $21a^2b^2 - 28ab$
5. Factorise $2xy^2 + 6x^2y + 8x^3y$
6. Simplify $2y + 4 \div 6y + 3 \times 4 - 5y$
7. Simplify $3 \div y + 2 \div y - 1$
8. Simplify $a^2 - 3ab \times 2a \div 6b + ab$

1.3 Revision of equations

(a) Simple equations

Problem 11. Solve $4 - 3x = 2x - 11$

Since $4 - 3x = 2x - 11$ then $4 + 11 = 2x + 3x$

i.e. $15 = 5x$ from which, $\boldsymbol{x} = \frac{15}{5} = \mathbf{3}$

Problem 12. Solve

$$4(2a - 3) - 2(a - 4) = 3(a - 3) - 1$$

Removing the brackets gives:

$$8a - 12 - 2a + 8 = 3a - 9 - 1$$

Rearranging gives:

$$8a - 2a - 3a = -9 - 1 + 12 - 8$$

i.e. $3a = -6$

and $\boldsymbol{a} = \frac{-6}{3} = \mathbf{-2}$

Problem 13. The reaction moment M of a cantilever carrying three, point loads is given by:

$$M = 3.5x + 2.0(x - 1.8) + 4.2(x - 2.6)$$

where x is the length of the cantilever in metres. If $M = 55.32$ kN m, calculate the value of x.

If $M = 3.5x + 2.0(x - 1.8) + 4.2(x - 2.6)$ and $M = 55.32$

then $55.32 = 3.5x + 2.0(x - 1.8) + 4.2(x - 2.6)$

i.e. $55.32 = 3.5x + 2.0x - 3.6 + 4.2x - 10.92$

and $55.32 + 3.6 + 10.92 = 9.7x$

i.e. $69.84 = 9.7x$

from which, **the length of the cantilever,**

$$\mathbf{x} = \frac{69.84}{9.7} = \mathbf{7.2\ m}$$

Problem 14. Solve $\frac{3}{x-2} = \frac{4}{3x+4}$

By 'cross-multiplying': $3(3x + 4) = 4(x - 2)$

Removing brackets gives: $9x + 12 = 4x - 8$

Rearranging gives: $9x - 4x = -8 - 12$

i.e. $5x = -20$

and $\boldsymbol{x} = \frac{-20}{5} = \mathbf{-4}$

Problem 15. Solve $\left(\frac{\sqrt{t} + 3}{\sqrt{t}}\right) = 2$

$$\sqrt{t}\left(\frac{\sqrt{t} + 3}{\sqrt{t}}\right) = 2\sqrt{t}$$

i.e. $\sqrt{t} + 3 = 2\sqrt{t}$

and $3 = 2\sqrt{t} - \sqrt{t}$

i.e. $3 = \sqrt{t}$

and $\mathbf{9} = \boldsymbol{t}$

(c) Transposition of formulae

Problem 16. Transpose the formula $v = u + \frac{ft}{m}$ to make f the subject.

$u + \frac{ft}{m} = v$ from which, $\frac{ft}{m} = v - u$

and $m\left(\frac{ft}{m}\right) = m(v - u)$

i.e. $ft = m(v - u)$

and $\boldsymbol{f} = \frac{\boldsymbol{m}}{\boldsymbol{t}}(\boldsymbol{v} - \boldsymbol{u})$

Problem 17. The impedance of an a.c. circuit is given by $Z = \sqrt{R^2 + X^2}$. Make the reactance X the subject.

$\sqrt{R^2 + X^2} = Z$ and squaring both sides gives

$R^2 + X^2 = Z^2$, from which,

$X^2 = Z^2 - R^2$ and **reactance** $\boldsymbol{X} = \sqrt{\boldsymbol{Z^2 - R^2}}$

Problem 18. Given that $\frac{D}{d} = \sqrt{\left(\frac{f+p}{f-p}\right)}$ express p in terms of D, d and f.

Rearranging gives: $\sqrt{\left(\frac{f+p}{f-p}\right)} = \frac{D}{d}$

Squaring both sides gives: $\frac{f+p}{f-p} = \frac{D^2}{d^2}$

'Cross-multiplying' gives:

$$d^2(f + p) = D^2(f - p)$$

Removing brackets gives:

$$d^2f + d^2p = D^2f - D^2p$$

Rearranging gives: $d^2p + D^2p = D^2f - d^2f$

Factorising gives: $p(d^2 + D^2) = f(D^2 - d^2)$

and $\boldsymbol{p} = \frac{\boldsymbol{f(D^2 - d^2)}}{\boldsymbol{(d^2 + D^2)}}$

Problem 19. Bernoulli's equation relates the flow velocity, v, the pressure, p, of the liquid and the height h of the liquid above some reference level.

Given two locations 1 and 2, the equation states:

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+h_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+h_2$$

where ρ is the density of the liquid. Rearrange the equation to make the velocity of the liquid at location 2, i.e. v_2, the subject.

Since $$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+h_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+h_2$$

then $$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+h_1-\frac{p_2}{\rho g}-h_2=\frac{v_2^2}{2g}$$

and $$2g\left(\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+h_1-\frac{p_2}{\rho g}-h_2\right)=v_2^2$$

from which, $$v_2=\sqrt{2g\left(\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+h_1-\frac{p_2}{\rho g}-h_2\right)}$$

i.e. **the velocity at location 2,**

$$\boldsymbol{v_2=\sqrt{2g\left(\frac{p_1}{\rho g}-\frac{p_2}{\rho g}+\frac{v_1^2}{2g}+h_1-h_2\right)}}$$

Now try the following Practice Exercise

Practice Exercise 3 Simple equations and transposition of formulae (Answers on page 863)

In problems 1 to 4 solve the equations

1. $3x-2-5x=2x-4$
2. $8+4(x-1)-5(x-3)=2(5-2x)$
3. $\dfrac{1}{3a-2}+\dfrac{1}{5a+3}=0$
4. $\dfrac{3\sqrt{t}}{1-\sqrt{t}}=-6$
5. Transpose $y=\dfrac{3(F-f)}{L}$ for f
6. Make l the subject of $t=2\pi\sqrt{\dfrac{l}{g}}$
7. Transpose $m=\dfrac{\mu L}{L+rCR}$ for L
8. Make r the subject of the formula $$\frac{x}{y}=\frac{1+r^2}{1-r^2}$$
9. Young's modulus, $E=\dfrac{\text{stress}, \sigma}{\text{strain}, \varepsilon}$ and strain, $\varepsilon=\dfrac{\text{length increase}, \delta}{\text{length}, \ell}$
For a 50 mm length of a steel bolt under tensile stress, its length increases by 1.25×10^{-4} m. If $E=2\times10^{11}$ N/m^2, calculate the stress.
10. The mass moment of inertia through the centre of gravity for a compound pendulum, I_G, is given by: $I_G=mk_G^2$. The value of the radius of gyration about G, k_G^2, may be determined from the frequency, f, of a compound pendulum, given by: $f=\dfrac{1}{2\pi}\sqrt{\dfrac{gh}{(k_G^2+h^2)}}$
Given that the distance $h=50$ mm, $g=9.81$ m/s^2 and the frequency of oscillation, $f=1.26$ Hz, calculate the mass moment of inertia I_G when $m=10.5$ kg.

(d) Simultaneous equations

Problem 20. Solve the simultaneous equations:

$$7x-2y=26 \quad (1)$$

$$6x+5y=29 \quad (2)$$

$5\times$ equation (1) gives:

$$35x-10y=130 \quad (3)$$

$2\times$ equation (2) gives:

$$12x+10y=58 \quad (4)$$

Equation (3) + equation (4) gives:

$$47x+0=188$$

from which, $$\boldsymbol{x}=\frac{188}{47}=\mathbf{4}$$

Substituting $x=4$ in equation (1) gives:

$$28-2y=26$$

from which, $28-26=2y$ and $\boldsymbol{y=1}$

Problem 21. Solve

$$\frac{x}{8}+\frac{5}{2}=y \quad (1)$$

$$11+\frac{y}{3}=3x \quad (2)$$

$8 \times$ equation (1) gives: $x + 20 = 8y$ (3)

$3 \times$ equation (2) gives: $33 + y = 9x$ (4)

i.e. $x - 8y = -20$ (5)

and $9x - y = 33$ (6)

$8 \times$ equation (6) gives: $72x - 8y = 264$ (7)

Equation (7) − equation (5) gives:

$$71x = 284$$

from which, $\boldsymbol{x} = \frac{284}{71} = \mathbf{4}$

Substituting $x = 4$ in equation (5) gives:

$$4 - 8y = -20$$

from which, $4 + 20 = 8y$ and $\boldsymbol{y} = \mathbf{3}$

(e) Quadratic equations

Problem 22. Solve the following equations by factorisation:

(a) $3x^2 - 11x - 4 = 0$

(b) $4x^2 + 8x + 3 = 0$

(a) The factors of $3x^2$ are $3x$ and x and these are placed in brackets thus:

$(3x \quad)(x \quad)$

The factors of -4 are $+1$ and -4 or -1 and $+4$, or -2 and $+2$. Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.,

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

Thus $(3x + 1)(x - 4) = 0$ hence

either $(3x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $(x - 4) = 0$ i.e. $\boldsymbol{x = 4}$

(b) $4x^2 + 8x + 3 = (2x + 3)(2x + 1)$

Thus $(2x + 3)(2x + 1) = 0$ hence

either $(2x + 3) = 0$ i.e. $\boldsymbol{x = -\frac{3}{2}}$

or $(2x + 1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{2}}$

Problem 23. The roots of a quadratic equation are $\frac{1}{3}$ and -2. Determine the equation in x.

If $\frac{1}{3}$ and -2 are the roots of a quadratic equation then,

$$(x - \frac{1}{3})(x + 2) = 0$$

i.e. $x^2 + 2x - \frac{1}{3}x - \frac{2}{3} = 0$

i.e. $x^2 + \frac{5}{3}x - \frac{2}{3} = 0$

or $\mathbf{3x^2 + 5x - 2 = 0}$

Problem 24. The stress, ρ, set up in a bar of length, ℓ, cross-sectional area, A, by a mass W, falling a distance h is given by the formula:

$$\rho^2 - \frac{2W}{A}\rho - \frac{2WEh}{A\ell} = 0$$

Given that $E = 12500$, $\ell = 110$, $h = 0.5$, $W = 2.25$ and $A = 3.20$, calculate the positive value of ρ correct to 3 significant figures.

Since $\rho^2 - \frac{2W}{A}\rho - \frac{2WEh}{A\ell} = 0$ and $E = 12500$, $\ell = 110$, $h = 0.5$, $W = 2.25$ and $A = 3.20$ then

$$\rho^2 - \frac{2(2.25)}{3.20}\rho - \frac{2(2.25)(12500)(0.5)}{(3.20)(110)} = 0$$

i.e.

$$\rho^2 - 1.40625\rho - 79.90057 = 0$$

Solving using the quadratic formula gives:

$$\rho = \frac{--1.40625 \pm \sqrt{(-1.40625)^2 - 4(1)(-79.90057)}}{2(1)}$$

$= 9.669$ or -8.263

Hence, correct to 3 significant figures, **the positive value of ρ is 9.67**

Now try the following Practice Exercise

Practice Exercise 4 Simultaneous and quadratic equations (Answers on page 863)

In problems 1 to 3, solve the simultaneous equations

1. $8x - 3y = 51$

   $3x + 4y = 14$

2. $5a = 1 - 3b$

   $2b + a + 4 = 0$

3. $\dfrac{x}{5} + \dfrac{2y}{3} = \dfrac{49}{15}$

   $\dfrac{3x}{7} - \dfrac{y}{2} + \dfrac{5}{7} = 0$

4. In an engineering scenario involving reciprocal motion, the following simultaneous equations resulted: $\omega\sqrt{(r^2 - 0.07^2)} = 8$ and $\omega\sqrt{(r^2 - 0.25^2)} = 2$
   Calculate the value of radius r and angular velocity ω, each correct to 3 significant figures.

5. Solve the following quadratic equations by factorisation:

   (a) $x^2 + 4x - 32 = 0$

   (b) $8x^2 + 2x - 15 = 0$

6. Determine the quadratic equation in x whose roots are 2 and -5

7. Solve the following quadratic equations, correct to 3 decimal places:

   (a) $2x^2 + 5x - 4 = 0$

   (b) $4t^2 - 11t + 3 = 0$

8. A point of contraflexure from the left-hand end of a 6 m beam is given by the value of x in the following equation:

   $$-x^2 + 11.25x - 22.5 = 0$$

   Determine the point of contraflexure.

9. The vertical height, h, and the horizontal distance travelled, x, of a projectile fired at an angle of 45° at an initial velocity,v_0, are related by the equation:

   $$h = x - \frac{gx^2}{v_0^2}$$

   If the projectile has an initial velocity of 120 m/s, calculate the values of x when the projectile is at a height of 200 m, assuming that $g = 9.81$ m/s^2. Give the answers correct to 3 significant figures.

1.4 Polynomial division

Before looking at long division in algebra let us revise long division with numbers (we may have forgotten, since calculators do the job for us!).

For example, $\dfrac{208}{16}$ is achieved as follows:

$$\begin{array}{r|l} & \;\;13 \\ \hline 16 & 208 \\ & 16 \\ \hline & \;\;48 \\ & \;\;48 \\ \hline & \;\;\cdot\cdot \end{array}$$

(1) 16 divided into 2 won't go

(2) 16 divided into 20 goes 1

(3) Put 1 above the zero

(4) Multiply 16 by 1 giving 16

(5) Subtract 16 from 20 giving 4

(6) Bring down the 8

(7) 16 divided into 48 goes 3 times

(8) Put the 3 above the 8

(9) $3 \times 16 = 48$

(10) $48 - 48 = 0$

Hence $\dfrac{208}{16} = \mathbf{13}$ exactly

Similarly, $\dfrac{172}{15}$ is laid out as follows:

$$\begin{array}{r|l} & \;\;11 \\ \hline 15 & 172 \\ & 15 \\ \hline & \;\;22 \\ & \;\;15 \\ \hline & \;\;\;\;7 \end{array}$$

Hence $\dfrac{172}{15} = 11$ remainder 7 or $11 + \dfrac{7}{15} = \mathbf{11\dfrac{7}{15}}$

Below are some examples of division in algebra, which in some respects is similar to long division with numbers.

(Note that a **polynomial** is an expression of the form

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$